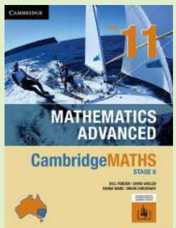
NORMANHURST BOYS HIGH SCHOOL

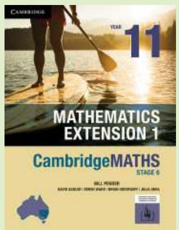
MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

Ⓐ ⓍⒾ **Introduction to Differentiation**

With references to





Name: ....

Initial version by H. Lam, November 2014. Last updated June 15, 2024.
Various corrections by students & members of the Mathematics Departments at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under (cc) CC BY 2.0.
Exercises from Section 7.2.1 on page 45 are taken from Dougherty and Gieringer (2014, Ch 4, p.306).
Picture credit: Weierstraß (page 7), Licensed under Public domain via Wikimedia Commons

Symbols used

- Beware! Heed warning.
- Mathematics Advanced content.
- Mathematics Extension 1 exclusive content.
- Literacy: note new word/phrase.
- Facts/formulae to memorise.
- On the course Reference Sheet.
- ICT usage
- Enrichment content. Broaden your knowledge!

Syllabus outcomes addressed

MA11-5 interprets the meaning of the derivative, determines the derivative of functions and applies these to solve simple practical problems

Syllabus subtopics

MA-C1 Introduction to Differentiation

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Cambridge Year 11, 3 Unit* (Pender, Sadler, Shea, & Ward, 1999), or *Cambridge Year 11 2 Unit* (Pender, Sadler, Shea, & Ward, 2009a) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

The derivative

Section 1

Limits and Continuity

Learning Goal(s)

Knowledge
What is a continuous function

Skills
Differentiate between continuous and discontinuous functions

Understanding
The features of graphs that are discontinuities

By the end of this section am I able to:

7.27 An intuitive approach to the concept of continuity

7.28 Distinguish between continuous and discontinuous functions, identifying key elements which distinguish each type of function

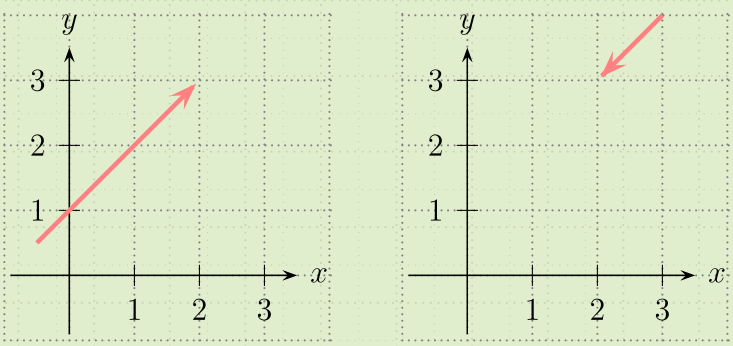
1.1 Limits

1.1.1 Definition

Example 1

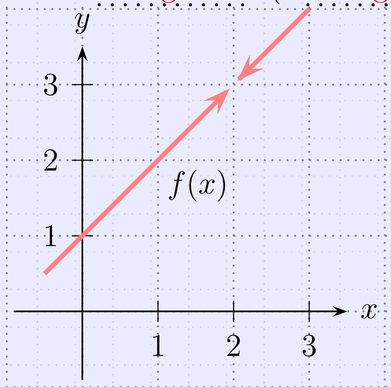
(a) Find the formula for $y = f(x)$.

(b) What function value (y value) are the arrowheads *approaching* in the diagram?



Definition 1

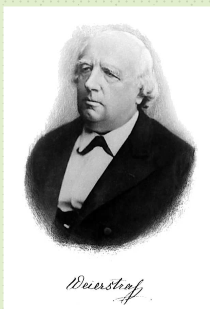
Limit (loose definition) A *full limit as $x \to x_0$* exists if the limit approaching from the left (“left hand limit”) is equal to the limit approaching from the right (“right hand limit”).



Notation:

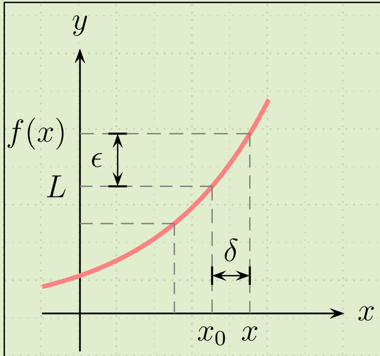
$$\lim_{x \to 2} f(x) = 3$$

History



Karl Weierstraß (1815-1897), cited as the “father of modern analysis”. Weierstraß left university without a degree, but studied and trained as a teacher.

Weierstraß’ interest lie in the *soundness* of calculus. Prior to his time, some definitions regarding the foundations of calculus were insufficiently rigorous. His work resulted in the formalisation of the definition of the *limit* (as well as the *continuity* of a function):



The limit of $f(x)$ as x approaches x_0 is L

$$\lim_{x \to x_0} f(x) = L$$

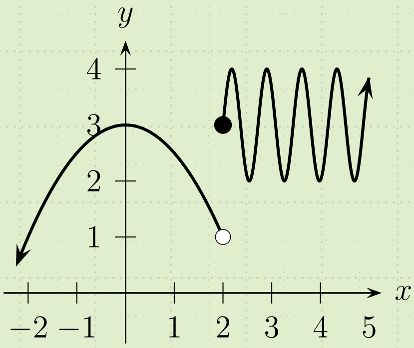
exists if and only if for every value $\epsilon > 0$, there exists another number $\delta > 0$ such that $|x - x_0| < \delta$ makes $|f(x) - L| < \epsilon$ true.

Further reading: Wikipedia

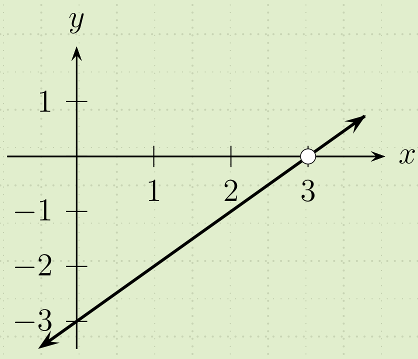
Example 2

State whether the following limits exist, giving a brief reason.

(a) $\lim\limits_{x \to 2} f(x)$



(b) $\lim\limits_{x \to 3} f(x)$



1.1.2 Evaluating simple limits

- If the function is "well behaved" at $x = a$, find its ....function........ ....value........ .
- Otherwise, use special techniques to evaluate.

Example 3

Evaluate $\lim\limits_{x \to -1} \left(x^3 - x^2\right)$.

Example 4

Evaluate $\lim\limits_{x \to -1} \dfrac{x^2 - 6x}{x}$.

Example 5

Evaluate $\displaystyle\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$.

Example 6

Evaluate $\displaystyle\lim_{x\to c}\frac{x^4-c^4}{x-c}$.

1.1.3 Special limits resulting in $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7

Sketch $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x^2}$, and hence, determine

(a) $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x}$ (b) $\displaystyle\lim_{x\to 0}\frac{1}{x}$ (c) $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x^2}$ (d) $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$

Laws/Results

- $\displaystyle\lim_{x\to\infty}\frac{1}{x}$ $= 0$
- $\displaystyle\lim_{x\to\infty}\frac{1}{x^2}$ $= 0$
- $\displaystyle\lim_{x\to-\infty}\frac{1}{x}$ $= 0$
- $\displaystyle\lim_{x\to-\infty}\frac{1}{x^2}$ $= 0$
- $\displaystyle\lim_{x\to 0}\frac{1}{x}$ not defined
- $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$ $= \infty$

Important note

⚠ LH/RH limit may not exist for the “full limit” to exist in these cases.

Theorem 1

When the limit of a rational function results in $\frac{0}{0}$ or $\frac{\infty}{\infty}$

- Factorise + cancel/simplify, or
- Divide numerator and denominators by the highest power of x to use the special limits.

Example 8

Evaluate the following limits:

Answer: (a) $-\frac{4}{7}$ (b) 1 (c) $\frac{5}{7}$ (d) ∞

(a) $\displaystyle\lim_{x\to 2}\frac{x^2-8x+12}{x^2+3x-10}$

(b) $\displaystyle\lim_{x\to \infty}\frac{x^2-8x+12}{x^2+3x-10}$

(c) $\displaystyle\lim_{x\to \infty}\frac{5x^2-x+9}{7x^2+2x+1}$

(d) $\displaystyle\lim_{x\to \infty}\frac{5x^3-x+9}{7x^2+2x+1}$

Further exercises (Legacy Textbooks)

(A) **Ex 9J**

- Q3, 4, 6, 7

(x1) **Ex 7I**

- Q1-5, 7-9

1.2 Continuity at $x = a$

Definition 2

A function is *continuous* at $x = a$ iff $f(a) = \lim_{x \to a} f(x)$,
i.e. its function value is equal to the full limit at that x coordinate.

Example 9

Determine whether the following functions are continuous at $x = 1$.

(a) $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.

(b) $f(x) = \begin{cases} \dfrac{x^2 - 5x + 4}{x - 1} & x \neq 1 \\ -3 & x = 1 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - 5x + 4}{x - 1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

(d) $f(x) = \begin{cases} 3^x & x < 1 \\ 7x - 4 & x = 1 \\ 5 - x^2 & x > 1 \end{cases}$

Example 10

If $g(x) = \dfrac{x^2 + 5x - 14}{x - 2}$ is defined for $x \neq 2$, what value must $g(2)$ take on for $g(x)$ continuous at $x = 2$?

Answer: 9

Example 11

Find the values of a and b if $Q(x)$ as shown, is to be continuous $\forall x$.

$$Q(x) = \begin{cases} x^2 & x \leq 1 \\ ax + b & 1 < x < 2 \\ x^2 - 5 & x \geq 2 \end{cases}$$

Answer: $a = -2$, $b = 3$

Figure 1.1 – Discontinuity of the Western Distributor (Sydney) near Anzac Bridge. Apple maps fail. Retrieved from www.smh.com.au galleries, 27/9/2012

Further exercises

A **Ex 8K**

- Q2, 3, 5

X1 **Ex 9K**

- Q1-8

Section 2

Finding the derivative ("first principles")

Learning Goal(s)

Knowledge
The first principles of differentiation

Skills
Find the derivative by first principles

Understanding
The relation between the derivative of a function $f(x)$ and its the rate of change

By the end of this section am I able to:

7.1 Interpret the derivative as the gradient of the tangent to the graph of $y = f(x)$ at a point x

7.2 Describe the gradient of a secant drawn through two nearby points on the graph of a continuous function as an approximation of the gradient of the tangent to the graph at those points, which improves in accuracy as the distance between the two points decreases

7.3 Interpret and use the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change of $f(x)$ or the gradient of a chord or secant of the graph $y = f(x)$

7.4 Examine the behaviour of the difference quotient $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$ as an informal introduction to the concept of a limit

7.5 Define the derivative $f'(x)$ from first principles, as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and use the notation for the derivative: $\frac{dy}{dx} = f'(x) = y'$ where $y = f(x)$

7.29 An intuitive approach to differentiability

Definition 3

Calculus: the mathematical study of change.

2.1 Goal

To find the gradient to the curve. For curves, the gradient at a particular x value can be found by

- Finding the tangent to the curve at that x value.
- Evaluating the gradient of the tangent.

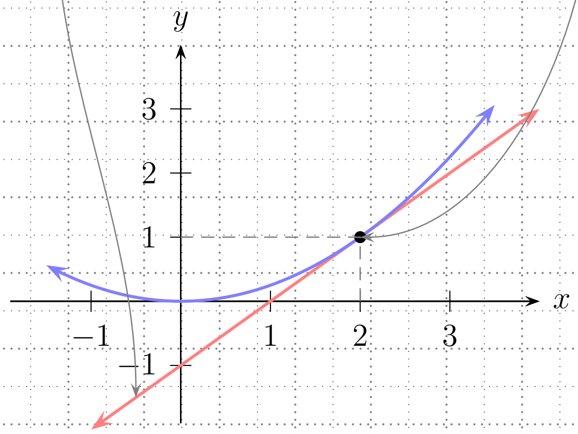


Figure 2.1 – Gradient of the curve

Problem: to find the equation of the tangent, two points are required. Use a secant instead to approximate.

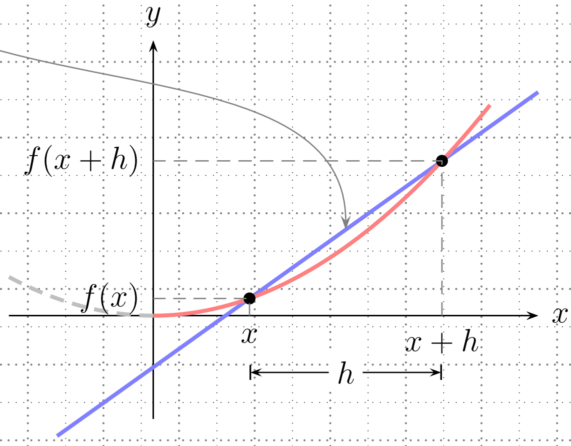


Figure 2.2 – Approximating the gradient of the curve

GeoGebra

Explore: `diff.ggb`

Definition 4

The *difference quotient*:

$$\frac{f(x+h)-f(x)}{h}$$

- As $h \to 0$, the secant will become the tangent.
- Gradient of the secant

$$m = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

- Take $h \to 0$:

Definition 5

The *gradient function* of $f(x)$, denoted $f'(x)$:

"$x+h$" : $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

"$u \to x$" : $f'(x) = \lim_{u\to x} \frac{f(u)-f(x)}{u-x}$

Other notation for the derivative:

$\frac{dy}{dx}$ y'

Noun Derivative, gradient function **Verb** Differentiate

2.2 Differentiability at $x = a$

Definition 6

A function $f(x)$ is *differentiable* at $x = a$ iff

1. It is ....continuous........ at $x = a$, and

2. $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists and is finite.

Example 12

For $y = |x|$,

(a) Rewrite as a piecewise defined function.

(b) Sketch the curve

(c) Find $f'(x)$, paying particular attention the point $x = 0$.

(d) Hence show that the derivative to $y = |x|$ does not exist at $x = 0$.

Laws/Results

Conclusion: a curve is *not* differentiable at $x = a$ when a

- ....discontinuity........
- ....cusp....

exists at $x = a$.

Laws/Results

At $x = a$:

........Differentiable........ $\longrightarrow$ ....Continuous........

$\longrightarrow$ Existence of ....Full.... ....limit....

2.2.1 Exercises

Use the definition of the derivative to find $f'(x)$. Question 5 onwards are more difficult.

1. $f(x) = 5 - 2x$

2. $f(x) = 10$

3. $f(x) = 2x^2 + 3$

4. $f(x) = 3x^2 - 5x + 9$

5. ⚠ $f(x) = \sqrt{x}^*$

6. ⚠ $f(x) = \dfrac{3}{x+2}$

7. ⚠ $f(x) = \sqrt{9 - 5x}$

8. ⚠ $f(x) = \dfrac{1}{x^2}$

9. ⚠ $f(x) = \dfrac{2}{\sqrt{x}}$

10. ⚠ $f(x) = x^{\frac{3}{2}\dagger}$

11. ⚠ $f(x) = 2x^3$

12. ⚠ $f(x) = \sqrt[3]{x+1}^{\ddagger}$

13. ⚠ $f(x) = x^4$

14. ⚠ $f(x) = \dfrac{x}{x+1}$

15. ⚠ $f(x) = \dfrac{x+1}{x-1}$

16. ⚠ $f(x) = x^{\frac{2}{3}}$

Answers

1. -2 **2.** 0 **3.** $4x$ **4.** $6x - 5$ **5.** $\frac{1}{2}x^{-\frac{1}{2}}$ **6.** $-3(x+2)^{-2}$ **7.** $-\frac{5}{2}(9-5x)^{-\frac{1}{2}}$ **8.** $-2x^{-3}$ **9.** $-x^{-\frac{3}{2}}$ **10.** $\frac{3}{2}x^{\frac{1}{2}}$ **11.** $6x^2$ **12.** $\frac{1}{3}(x+1)^{-\frac{2}{3}}$ **13.** $4x^3$ **14.** $(x+1)^{-2}$ **15.** $-2(x-1)^{-2}$ **16.** $\frac{2}{3}x^{-\frac{1}{3}}$

Further exercises

(A) **Ex 8B**
- Q1-10
- Q14

(X1) **Ex 9B**
- Q3-9

(X1) **Ex 9L**
- Q1-8

* *Hint:* use difference of squares
† *Hint:* rewrite as $\sqrt{x^3}$
‡ *Hint:* use difference of cubes

Section 3

Finding the derivative (shortcut)

Learning Goal(s)

Knowledge
The shortcut method for derivatives

Skills
Find the derivative efficiently by applying the shortcut method

Understanding
How to differentiate sums and differences of terms with coefficients

By the end of this section am I able to:

7.8 Use the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ for all real values of n

7.9 Differentiate a constant multiple of a function and the sum or difference of two functions

3.1 Polynomial-like terms

Theorem 2

If $f(x) = x^n$, where $n \in \mathbb{R}$, then

$$f'(x) = nx^{n-1}$$

Laws/Results

Rules for differentiating polynomial-like terms:

1. Derivative of a sum is the sum of derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

2. Coefficients are "left alone"

$$\frac{d}{dx}[af(x)] = a\frac{d}{dx}[f(x)]$$

Important note

Use this shortcut hereforth unless the question asks for **first principles**!

Example 13

Find $\dfrac{dy}{dx}$ for the following:

1. $y = 6x - 4$

2. $y = 2x^3 - 4x$

3. $y = x^3 - 5x^2 + 2x - 1$

3.2 Index laws

Example 14

Find the derivatives of the following, first by rewriting in index form.

1. $\dfrac{16}{x^2} + \dfrac{16}{x^3}$

2. $\dfrac{5}{12x}$

3. $24\sqrt{x}$

4. $24\sqrt[3]{x}$

5. $\sqrt[3]{x^2}$

6. $2x\sqrt{x}$

7. $\sqrt{5x}$

8. $\dfrac{1}{\sqrt{x}}$

9. $\dfrac{6}{\sqrt{x}}$

10. $\dfrac{5}{x\sqrt{x}}$

⚠ Be careful with your notation!

3.3 Expansions/partial fractions

Example 15

Find the derivatives of the following by first expanding brackets.

1. $3x^2\left(x-2x^3\right)$

2. $(2x-1)(2-3x)$

3. $3x^{-2}-2x^{-1}$

4. $x^{-2}\left(x^2-x+1\right)$

Example 16

Find the derivatives of the following by "splitting" the fractions.

1. $\dfrac{3x^4-2x^3}{x^2}$

2. $\dfrac{x^2+2x+1}{x^2}$

3. $\dfrac{4x+5\sqrt{x}}{\sqrt{x}}$

4. $\dfrac{2x^2\sqrt{x}+3x\sqrt{x}}{x}$

Further exercises

A **Ex 8C**
- Q1-19

A **Ex 8E**
- Q1-5

A **Ex 8F**
- Q2-5, 10-11

X1 **Ex 9C**
- Q1-3

X1 **Ex 9D**
- Q1, 3-5

Section 4

Tangents and normals

Learning Goal(s)

Knowledge
The relation between the first derivative and tangents

Skills
Finding the equation of tangents and normals to a curve

Understanding
The first derivative as a gradient function

By the end of this section am I able to:

7.10 Use the derivative in a variety of contexts, including finding the equation of a tangent or normal to a graph of a power function at a given point

4.1 Definitions

Definition 7

A *tangent* to the curve at $x = a$ touches the curve at that point.

Definition 8

The *normal* to the curve at $x = a$ is the line that is perpendicular to the tangent at $x = a$.

Diagram:

Important note

Use coordinate geometry methods to solve problems related to tangents and normals.

4.2 Examples

Example 17

Given that $f(x) = x^3 - 3x$,

(a) Find the equation of the tangent and normal at $P(2, 2)$.

(b) Find the points on the curve where the tangent is horizontal.

Answer: (a) Tangent: $y = 9x - 16$, Normal: $y = -\frac{1}{9}x + \frac{20}{9}$ (b) $(1, -2)$ and $(-1, 2)$

Example 18

Find the points on the graph of $f(x) = x + \dfrac{1}{x}$ where

(a) The tangent is horizontal.

(b) The normal has gradient of -2.

(c) The tangent has an angle of inclination of $45°$.

Answer: (a) $(1, 2)$ and $(-1, -2)$ (b) $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$, and $\left(-\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$

Example 19

[Ex 7D Q17] Show that the line $x + y + 2 = 0$ is tangent to $y = x^3 - 4x$, and find the point of contact.

Hint: Find the equations of the tangents parallel to $x + y + 2 = 0$, and show that one of them is this very line.

Important note

Be careful with your notation! The following are two very different statements:

- If $\dfrac{dy}{dx} = 4x^3 - 2x$, find the value of $\dfrac{dy}{dx}$ at $x = -1$
  - Differential notation:

    $$\frac{dy}{dx} = 4x^3 - 2x\Big|_{x=-1}$$

  - Function notation:

    $$f'(-1) = 4x^3 - 2x$$

- If $\dfrac{dy}{dx} = 4x^3 - 2x$, find the value of x where the gradient of the tangent is -1.
  - Differential notation:

    $$\frac{dy}{dx} = 4x^3 - 2x = -1$$

  - Function notation:

    $$f'(x) = 4x^3 - 2x = -1$$

Further exercises

(A) **Ex 8D**
- Q1-18

(A) **Ex 8E**
- Q6-12
- ⚠ Q15

(A) **Ex 8F**
- Q6-9

(X1) **Ex 9C**
- Q4-16

(X1) **Ex 9D**
- Q9-12, Q14-20

Section 5

Other rules for finding the derivative

Learning Goal(s)

Knowledge
What is the product, quotient and chain rule

Skills
Identifying u and v to apply the product, quotient and chain rule

Understanding
Which rule to apply when differentiating more complex functions

By the end of this section am I able to:

7.12 Understand and use the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\frac{f(x)}{g(x)}$ and $f(g(x))$ where $f(x)$ and $g(x)$ are functions

7.13 Further work with the chain rule

5.1 Chain rule

Theorem 3

If $y = f\big(u(x)\big)$, then $f'(x) = f'(u) \times u'(x)$.

Alternatively,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Important note

- Look for an outer and inner function.

Example 20

Use the chain rule to differentiate: **Answer: 1.** $12\left(x^2+1\right)^5$ **2.** $105(3x+4)^4$

1. $\left(x^2+1\right)^6$ **2.** $7\left(3x+4\right)^5$ **3.** $(ax+b)^n$ **4.** $\sqrt{25-x^2}$

Example 21

[**Ex 7E Q11**] Find the values of a and b if the parabola $y = a(x+b)^2 - 8$:

(a) has tangent $y = 2x$ at the point $P(4,8)$

(b) has a common tangent with $y = 2 - x^2$ at the point $A(1,1)$.

Answer: (a) $a = \frac{1}{16}$, $b = 12$ (b) $a = \frac{1}{9}$, $b = -10$

Further exercises

Ⓐ **Ex 8G**

- Q2-17, 19

(x1) **Ex 9E**

- Q2-12

5.2 Product rule

Theorem 4

If $y = u(x)v(x)$, then $f'(x) = u(x)v'(x) + v(x)u'(x)$.

Alternatively,

$$\frac{dy}{dx} = uv' + vu'$$

Important note

- Look for a **product** of two functions
  - ⚠ ....Constants........ are *not* functions in this instance!
- Write down explicitly, the functions represented by u and v!

Example 22

Differentiate each function, expressing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

1. $y = x(x-10)^4$ **2.** $y = x^2(3x+2)^3$ **3.** $y = x\sqrt{x+3}$

Answer: **1.** $5(x-10)^3(x-2)$, $x = 2, 10$ **2.** $x(3x+2)^2(15x+4)$, $x = 0, -\frac{2}{3}, -\frac{4}{15}$ **3.** $\frac{3(x+2)}{2\sqrt{x+3}}$, $x = -2$

Example 23

Differentiate:

1. $f(x) = (3x+2)(2x^2-3x+4)$ **2.** $f(x) = \left(x^3+5x^2-3\right)\left(x^2+1\right)^5$

Answer: 1. $f'(x) = 18x^2 - 10x + 6$ **2.** $f'(x) = x\left(13x^3+60x^2+3x-20\right)\left(x^2+1\right)^4$

Further exercises

(A) Ex 8H
- Q2-16

(X1) Ex 9H
- Q2-12

5.3 Quotient rule

Theorem 5

If $y = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$.

Alternatively,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

Important note

- Look for a **quotient** of two functions
- Write down explicitly, the functions represented by u and v!

Example 24

Differentiate, stating when the derivative is zero:

1. $\dfrac{2x+1}{2x-1}$

2. $\dfrac{\sqrt{x+1}}{x}$

Answer: 1. $-\frac{4}{(2x-1)^2}$ **2.** $\frac{-x-2}{2x^2\sqrt{x+1}}$

Example 25

Differentiate and fully simplify: $\dfrac{(3x+4)^5}{(2x-5)^3}$.

Answer: $\frac{3(3x+4)^4(4x-33)}{(2x-5)^4}$

Further exercises

Ⓐ Ex 8I

- Q2-9

X1 Ex 9I

- Q2-8, 10-15

Part II

The function and its subsequent derivatives

Section 6

Values of $f'(x)$

6.1 Increasing, decreasing, stationary at a point

Learning Goal(s)

Knowledge
The first derivative represents the rate of change of a function

Skills
Determine when a function is increasing and decreasing

Understanding
The relation between the first derivative and the behaviour of its graph

By the end of this section am I able to:

7.16 Understand the concept of the derivative as a function

7.17 Sketch the derivative function (or gradient function) for a given graph of a function, without the use of algebraic techniques

Definition 9

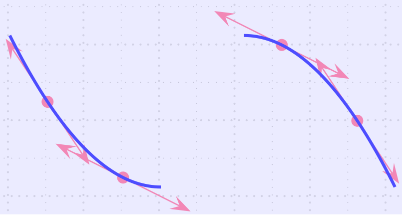
A function $f(x)$ is

- *increasing* at $x = a$ if its ......derivative...... is ......positive...... at that point, i.e.

$$\frac{dy}{dx} \ldots > \ldots 0$$

- *decreasing* at $x = a$ if its ......derivative...... is ......negative...... at that point, i.e.

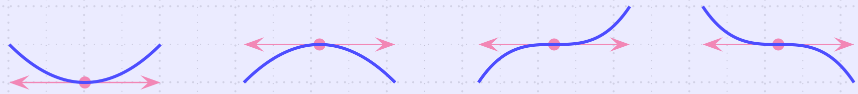
$$\frac{dy}{dx} \ldots < \ldots 0$$



Definition 10

A function $f(x)$ is *stationary* at the point $x = a$ if its derivative is zero at that point, i.e.

$$\frac{dy}{dx} = 0$$



Example 26

Find where $y = x^3 - 4x$ is decreasing.

Answer: $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

Example 27

[Ex 10A Q4]

(a) Find $f'(x)$ for the function $f(x) = x^3 - 3x^2 + 5$.

(b) For what values of x is:

i $f'(x) > 0$ ii $f'(x) < 0$ iii $f'(x) = 0$

(c) Evaluate $f(0)$ and $f(2)$, then, by interpreting these results geometrically, sketch a graph of $y = f(x)$.

Example 28

[**Ex 10A Q10(b)**] By finding $f'(x)$ show that $f(x) = \dfrac{x^3}{x^2+1}$ is increasing for all x, apart from $x = 0$ where it is stationary.

Example 29

[**Ex 10A Q15(c)**] Sketch the graph of the continuous curve suggested by the properties below:

- $f(x)$ is odd
- $f'(x) > 0$ for $x > 1$
- $f(3) = 0$ and $f'(1) = 0$
- $f'(x) < 0$ for $0 \leq x < 1$

Example 30

For what values is $y = \dfrac{x^2}{2x^2 + x + 1}$ decreasing?

Answer: $-2 < x < 0$

Further exercises

Ⓐ **Ex 3A (Y12 textbook)**

- Q1-17, 19-23

(X1) **Ex 4A (Y12 textbook)**

- Q1-15

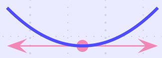
6.2 Stationary & turning points

Definition 11

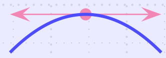
A **turning point** is a stationary point where the sign of the derivative changes.

Types of turning points

- local minimum



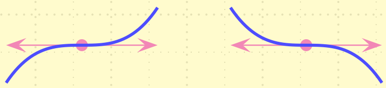
- local maximum



Laws/Results

Other stationary points:

- horizontal point of inflexion



Example 31

Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, determine their nature, and sketch the curve.

Example 32

The graph of the cubic $f(x) = x^3 + ax^2 + bx + c$ passes through the origin and has a stationary point at $A(2, 2)$. Find a, b and c. **Answer:** $a = -\frac{9}{2}$, $b = 6$, $c = 0$

Further exercises

Ⓐ Ex 3B (Y12 textbook)

- Q2-15

(X1) Ex 4B (Y12 textbook)

- Q5-17

Section 7

The second derivative and concavity of a curve

Learning Goal(s)

Knowledge
The relation between the first derivative and tangents

Skills
Finding the equation of tangents and normals to a curve

Understanding
The first derivative as a gradient function

By the end of this section am I able to:

7.11 Develop the concept of the second and higher order derivatives

7.14 Parametric differentiation via the chain rule: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

7.1 Finding the second derivative

Definition 12

The second derivative of a function $f(x)$ is found by ....differentiating........ $f(x)$, then ....differentiating........ $f'(x)$.

Important note

If the first derivative measures the ....change.... of $f(x)$, the second derivative measures the ....change.... of the ....change.... .

Notation: denoted in one of the following ways:

- $f''(x)$
- $\frac{d^2y}{dx^2}$
- $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

Example 33

[Ex 10D Q7(c)] Find the first and second derivatives of $\dfrac{x}{1+x^2}$.

Example 34

Find the first and second derivatives of $f(x) = x^2\sqrt{x}$.

Example 35

Find the first, second and third derivatives of x^n. Hence find the n-th and $(n+1)$-th derivatives of x^n.

Further exercises

(A) Ex 3C (Y12 textbook)
- Q1-12

(x1) Ex 4D
- Q1-12

7.2 (X1) Parametric differentiation

Theorem 6

Derivatives of parametric equations require the chain rule

First derivative

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

Example 36

(Fitzpatrick & Aus, 2019, Ex 7.5 Q4)

(a) If $x = 4t$ and $y = 2t^2$, find the expression for $\frac{dy}{dx}$ in terms of t.

(b) Hence find the expression for $\frac{dy}{dx}$ in terms of x.

Answer: (a) $\frac{dy}{dx} = t$ (b) $\frac{dy}{dx} = \frac{x}{4}$

Example 37

[**Ex 9E Q14**] Find the tangent to the curve at the point where $t = 3$:

$$x = 5t^2 \ , \ y = 10t$$

Answer: $y = \frac{1}{3}x + 15$

Example 38

(Fitzpatrick & Aus, 2019, Ex 7.5 Q6)

(a) If $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1-t^2}{1+t^2}$, find the expression for $\dfrac{dy}{dx}$ in terms of t.

(b) Hence find $\dfrac{d^2y}{dx^2}$ as a function of t.

Answer: (a) $\frac{dy}{dx} = \frac{2t}{t^2-1}$ (b) $\frac{d^2y}{dx^2} = \frac{(t^2+1)^3}{(t^2-1)^3}$

Example 39

(Fitzpatrick & Aus, 2019, Ex 7.5 Q7)

(a) If $x = t^2 + 4t$ and $y = 3t + t^3$, find the expression for $\frac{dy}{dx}$ in terms of t.

(b) If $\dfrac{dy}{dx} = 1$, find the values of x.

(c) Find $\dfrac{d^2y}{dx^2}$ as a function of t.

Answer: (a) $\frac{dy}{dx} = \frac{3(1+t^2)}{2(t+2)}$ (b) $x = -\frac{11}{9}$ or 5 (c) $\frac{d^2y}{dx^2} = \frac{3(t^2+4t-1)}{4(t+2)^3}$

7.2.1 Exercises

(Fitzpatrick & Aus, 2019, Chapter Review 7)

1. Given $x = t^2 - 1$ and $y = t^3$, find as a function of t:

(a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$

2. Given $x = 40t$ and $y = 56t - 16t^2$, find the expression for $\dfrac{dy}{dx}$.

3. (a) If $x = 2\left(t + \dfrac{1}{t}\right)$, $y = 2\left(t - \dfrac{1}{t}\right)$, find an expression for $\dfrac{dy}{dx}$ in terms of t.

(b) Find $\dfrac{d^2y}{dx^2}$ as a function of t.

(Pender et al., 1999, Ex 7K)

4. (a) Use parametric differentiation to differentiate the function defined by $x = t + \dfrac{1}{t}$ and $y = t - \dfrac{1}{t}$, and find the tangent and normal at the point T where $t = 2$.

(b) Eliminate t from these equations, and use ~~implicit~~ parametric differentiation to find the gradient of the curve at the same point T. [HINT: Square x and y and subtract.]

Answers

1. (a) $\frac{dy}{dx} = \frac{3t}{2}$ (b) $\frac{d^2y}{dx^2} = \frac{3}{4t}$ **2.** $\frac{dy}{dx} = \frac{7-4t}{5}$ **3.** (a) $\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$ (b) $\frac{d^2y}{dx^2} = \frac{-2t^3}{(t^2-1)^3}$ **4.** (a) $y' = \frac{t^2+1}{t^2-1}$, tangent: $5x - 3y = 8$, normal: $3x + 5y = 15$ (b) $x^2 - y^2 = 4$

Further exercises

X1 **Ex 9E**	X1 **Ex 9F**	X1 **Ex 9I**
• Q13-14	• Q14-15	• Q9

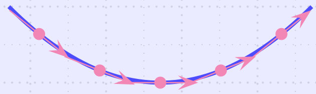
7.3 Concavity

Definition 13

A curve concaves

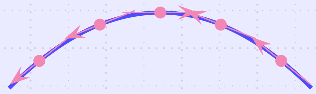
- *up* at a point $x = a$ when its second derivative at that point is ....positive..........

$$f''(a) \ldots > 0 \ldots$$



- *down* at a point $x = a$ when its second derivative at that point is ....negative..........

$$f''(a) \ldots < 0 \ldots$$



Laws/Results

Consequences for stationary points A stationary point coinciding where the curve concaves:

- up: .....local..... ....minimum..........
- down: .....local..... ....maximum..........

7.3.1 Change in concavity

Definition 14

A *point of inflexion* occurs on when the concavity of the curve changes, from concave up to concave down and vice versa. At all points of inflexion, $f''(x) = 0$.

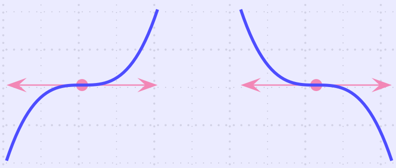
Important note

- Geometrically, a point of inflexion is a point where the tangent crosses the curve, i.e. the curve must 'curl away' from the tangent on opposite sides of the tangent.
- Example where $f''(x) = 0$ but does not give a point of inflexion – $f(x) = x^4$ around $x = 0$. Concavity does not change here!

Draw four examples of points of inflexion.

Definition 15

A *horizontal point of inflexion* occurs when the derivative $f'(x) = 0$ also where the concavity changes.



Example 40

[2007 HSC Q6/Ex 10E] Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflexion, examine the concavity over the whole domain, and sketch the curve.

Example 41

For what values of b is the graph of the quartic $f(x) = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down at the point where $x = 2$?

Answer: $b > \frac{29}{6}$

Example 42

[2013 2U HSC Q12] (2 marks) The cubic $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $x = p$.

Show that $p = -\dfrac{b}{3a}$.

Example 43

Using the following information, sketch the graph of $y = f(x)$:

- $f(0) = 0$
- $f'(x) > 0$ for all x
- $f''(0) = 0$
- $f''(x) > 0$ when $x < 0$
- $f''(x) < 0$ when $x > 0$

Example 44

[**2012 HSC Q14**] A function is given by $f(x) = 3x^4 + 4x^3 - 12x^2$.

i. Find the coordinates of the stationary points of $f(x)$ and determine their nature. **3**

ii. Hence, sketch the graph of $y = f(x)$ showing the stationary points. **2**

iii. For what values of x is the function increasing? **1**

iv. For what values of k will $3x^4 + 4x^3 - 12x^2 + k = 0$ have no solution? **1**

7.4 Graphs of successive derivatives

Laws/Results

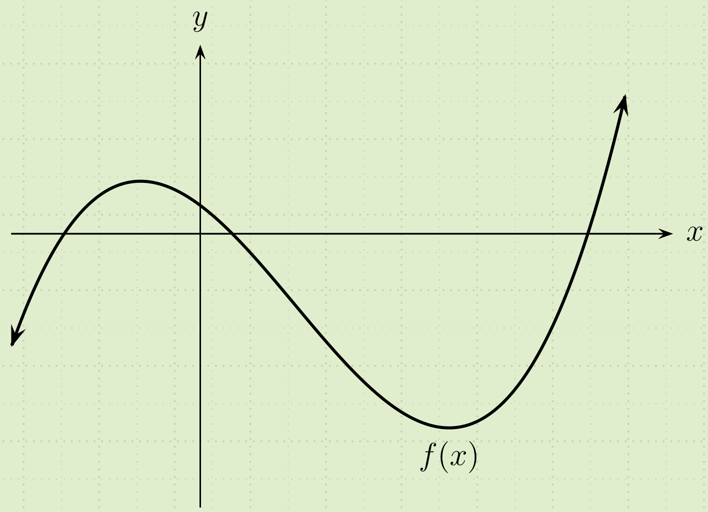
Relationship between $f(x)$, $f'(x)$ and $f''(x)$

Feature	**Significance of feature w.r.t. $f(x)$** $f(x)$	$f'(x)$	$f''(x)$
Zero crossing			
Turning point			N/A
Pt of inflexion of f			$f''(x) = 0$
HPOI of f			$f''(x) = 0$

Calculus grapher http://phet.colorado.edu/en/simulation/calculus-grapher

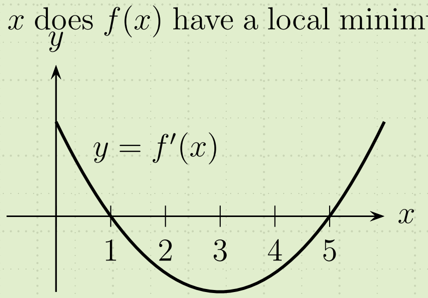
Example 45

The diagram shows the graph of a function $y = f(x)$. Sketch its derivative $f'(x)$ on the same set of axes.



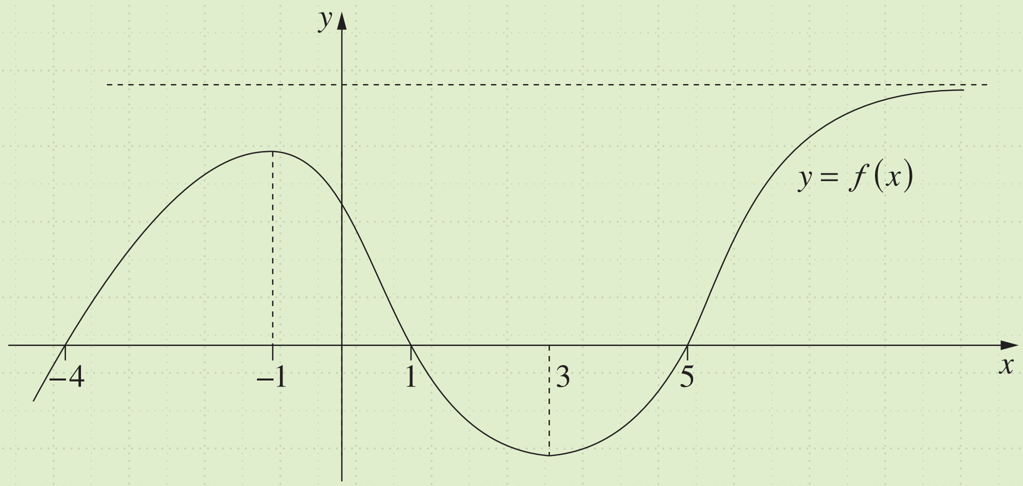
Example 46

[1999 HSC Q8] The diagram shows the graph of the gradient function of the curve $y = f(x)$. For what value of x does $f(x)$ have a local minimum? Justify your answer.



Example 47

[2009 HSC Q8] The diagram shows the graph of a function $y = f(x)$.



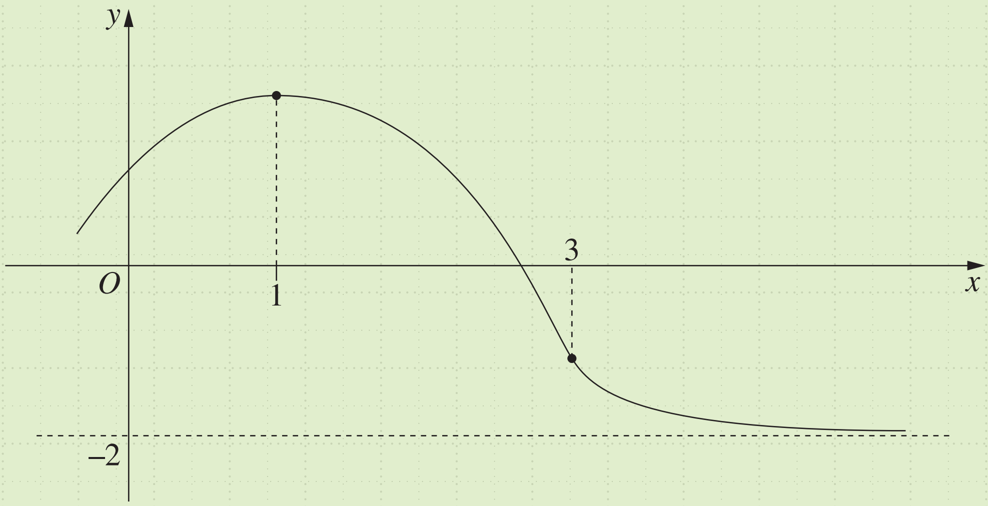
i. For which values of x is the derivative, $f'(x)$, negative? **1**

ii. What happens to $f'(x)$ for large values of x? **1**

iii. Sketch the graph of $y = f'(x)$. **2**

Example 48

[2011 HSC Q9] (3 marks) The graph $y = f(x)$ in the diagram has a stationary point when $x = 1$, a point of inflexion when $x = 3$, and a horizontal asymptote $y = -2$.



Sketch the graph of $y = f'(x)$, clearly indicating its features at $x = 1$ and at $x = 3$, and the shape of the graph as $x \to \infty$.

Further exercises

(A) Ex 3D (Y12 textbook)
- Q1-6
- Q7-19 odd

(X1) Ex 4E (Y12 textbook)
- Q1-23 odd

Part III

Applications

Section 8

Basic rates of change

Learning Goal(s)

Knowledge
The difference between instantaneous and average rate of change

Skills
Solve problems involving rates of change using derivatives

Understanding
The real-life implications of the first and second derivative functions

By the end of this section am I able to:

7.18 Consider average rate of change and relate this to instantaneous rate of change

7.19 Interpret and use the derivative at a point as the instantaneous rate of change of a function at that point

7.20 Calculate derivatives of power functions to solve problems, including finding an instantaneous rate of change of a function in both real life and abstract situations

7.23 Interpret the derivative as a measure of instantaneous rate of change.

7.24 Describe the behaviour of a function and its tangent at a point, using language including increasing, decreasing, constant, stationary, increasing at an increasing rate

7.26 Solve a variety of problems involving (simple) rates of change

8.1 Instantaneous vs Average Rate of Change

Definition 16

If a quantity is measured by Q, then the

- Instantaneous rate of change is $\dfrac{dQ}{dt}$, i.e. gradient of tangent.
- Average rate of change is $\dfrac{Q_2 - Q_1}{t_2 - t_1}$, where Q_1, Q_2 are the quantities at the start and finish of the times measured, and the difference between t_1 and t_2 is the time elapsed, i.e. gradient of secant.

Draw example of how to measure average and instantaneous rates of change.

Example 49

If $f(t) = t^2 - 2t + 4$, find

(a) The average rate of change between $t = 2$ and $t = 4$.

(b) A new function that describes the rate of change

(c) The instantaneous rate of change when $t = 4$.

Answer: (a) 4 (b) $2t - 2$ (c) 6

Example 50

A javelin is thrown so that its height, h metres above the ground, is given by the rule: $h(t) = 20t - 5t^2 + 2$, where t represents time in seconds.

(a) Find the rate of change of the height at any time, t.

(b) Find the rate of change of the height when

i. $t = 1$ ii. $t = 2$ iii. $t = 3$

(c) Briefly explain why the rate of change is initially positive, then zero, and then negative over the first 3 seconds.

(d) Find the rate of change of the height when the javelin first reaches a height of 17 metres.

Answer: (a) $20 - 10t$ (b) i. $10\,\text{ms}^{-1}$ ii. $0\,\text{ms}^{-1}$ iii. $-10\,\text{ms}^{-1}$ (c) Explain. (d) $10\,\text{ms}^{-1}$

Example 51

(Pender, Sadler, Shea, & Ward, 2009b, p.260) A cockroach plague hit the suburb of Berrawong last year, but was gradually brought under control. The council estimated that the cockroach population P, in millions, t months after 1st January, was given by

$$P = 7 + 6t - t^2$$

(a) Differentiate to find the rate of change $\dfrac{dP}{dt}$ of the cockroach population.

(b) Find the cockroach population on 1st January and the rate at which the population was increasing at that time.

(c) When did the council manage to stop the cockroach population increasing any further, and what was the population then?

(d) When were the cockroaches finally eliminated?

(e) What was the average rate of increase in the population from 1st January to 1st April?

Example 52

A water tank is being emptied and the quantity of water, Q litres, remaining in the tank at any time, t minutes, after it starts to empty is given by:

$$Q(t) = 1000(20 - t)^2$$

(a) At what rate is the tank being emptied at any time t?

(b) How long does it take to empty the tank?

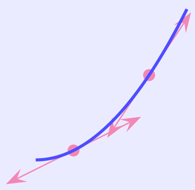
(c) At what time is the water flowing out at the rate of 20 000 litres per minute?

(d) What is the average rate at which the water flows out in the first 5 minutes?

8.1.1 Increase or decreasing, at an increasing/decreasing rate

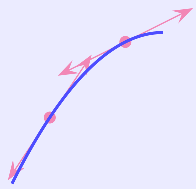
Strange English grammar ahead!

Definition 17



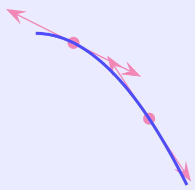
- Increasing at an ......increasing...... rate
- Observation: $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} > 0$
- The rate of increase, is ......increasing......

Definition 18



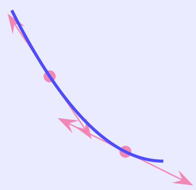
- Increasing at a ......decreasing...... rate
- Observation: $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$
- The rate of increase, is ......decreasing......

Definition 19



- Decreasing at an ......increasing...... rate
- Observation: $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} < 0$
- The rate of decrease, is ......increasing......

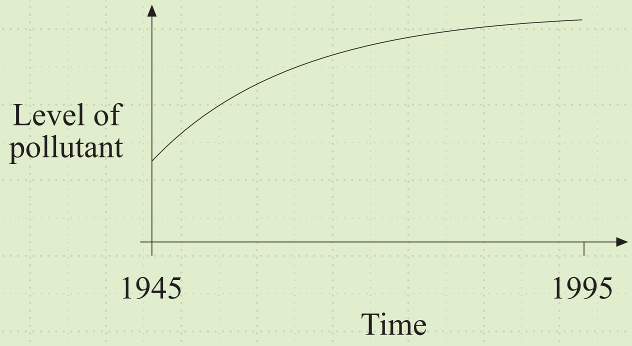
Definition 20



- Decreasing at a ......decreasing...... rate
- Observation: $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$
- The rate of decrease, is ......decreasing......

Example 53

[**1995 2U HSC**] The graph shows the levels of a pollutant in the atmosphere over the past 50 years.



Describe briefly how the level of this pollutant has changed over this period of time. Include mention of the rate of change.

Example 54

[**1997 2U HSC**] The rate of inflation measures the rate of change in prices. Between January 1996 and December 1996, prices were rising but the rate of inflation was falling. Draw a graph of prices as a function of time that fits this description.

Example 55

[**2000 2U HSC**] The number N of students logged onto a website at any time over a five-hour period is approximated by the formula

$$N = 175 + 18t^2 - t^4 \quad 0 \leq t \leq 5$$

(i) What was the initial number of students logged onto the website? **1**

(ii) How many students were logged onto the website at the end of the five hours? **2**

(iii) What was the maximum number of students logged onto the website? **2**

(iv) When were the students logging onto the website most rapidly? **2**

(v) Sketch the curve $N = 175 + 18t^2 - t^4$ for $0 \leq t \leq 5$. **2**

Example 56

[2002 2U Q7] A cooler, which is initially full, is drained so that at time t seconds the volume of water V, in litres, is given by

$$V = 25\left(1 - \frac{t}{60}\right)^2 \text{ for } 0 \leq t \leq 60$$

(i) How much water was initially in the cooler? **1**

(ii) After how many seconds was the cooler one-quarter full? **2**

(iii) At what rate was the water draining out when the cooler was one-quarter full? **2**

Example 57

[2000 3U Q7] ⚠ The amount of fuel F in litres required *per hour* to propel a plane in level flight at constant speed u km/h is given by

$$F = Au^3 + \frac{B}{u}$$

where A and B are positive constants.

(i) Show that a pilot wishing to remain in level flight for as long a period as possible should fly at **2**

$$\left(\frac{B}{3A}\right)^{\frac{1}{4}} \text{ km/h}$$

(ii) Show that a pilot wishing to fly as far as possible in level flight should fly approximately 32% faster than the speed given in part (i). **2**

EXERCISE 9A Rates of change

WORKED Example 1

1 If $f(x) = x^2 + 5x + 15$ find:

a the average rate of change between $x = 3$ and $x = 5$

b a new function that describes the rate of change

c the (instantaneous) rate of change when $x = 5$.

2 A balloon is inflated so that its volume, V cm^3, at any time, t seconds later is:

$$V = -\tfrac{8}{5}t^3 + 24t^2,\ t \in [0, 10]$$

a What is the volume of the balloon when:
i $t = 0$? **ii** $t = 10$?

b Hence, find the average rate of change between $t = 0$ and $t = 10$.

c Find the rate of change of volume when
i $t = 0$ **ii** $t = 5$ **iii** $t = 10$.

3 multiple choice

The average rate of change between $x = 1$ and $x = 3$ for the function $y = x^2 + 3x + 5$ is:

A 1 **B** 9 **C** 5 **D** 3 **E** 7

4 multiple choice

The instantaneous rate of change of the function $f(x) = x^3 - 3x^2 + 4x$, when $x = -2$ is:

A 2 **B** -2 **C** 28 **D** 3 **E** 12

5 multiple choice

If the rate of change of a function is described by $\dfrac{dy}{dx} = 2x^2 - 7x$, then the function could be:

A $y = 6x^3 - 14x$ **B** $y = \tfrac{2}{3}x^3 - 7x$ **C** $y = \tfrac{2}{3}x^3 - \tfrac{7}{2}x^2 + 5$

D $y = x^3 - \tfrac{7}{2}x^2 + 2$ **E** $2x^2 - 7x + 5$

WORKED Example 2

6 In a baseball game the ball is hit so that its height above the ground, h metres, is

$$h(t) = 1 + 18t - 3t^2$$

t seconds after being struck.

a Find the rate of change, $h'(t)$.

b Calculate the rate of change of height after:
i 2 seconds
ii 3 seconds
iii 4 seconds.

c What happens when $t = 3$ seconds?

d Find the rate of change of height when the ball first reaches a height of 16 metres.

7 The position, x metres, of a lift (above ground level) at any time, t seconds, is given by:

$$x(t) = -2t^2 + 40t$$

a Find the rate of change of displacement (velocity) at any time, t.
b Find the rate of change when:
i $t = 5$ **ii** $t = 9$ **iii** $t = 11$.
c What happened between $t = 9$ and $t = 11$?
d When and where is the rate of change zero?

8 The number of seats, N, occupied in a soccer stadium t hours after the gates are opened is given by:

$$N = 500t^2 + 3500t,\ t \in [0, 5]$$

a Find N when:
i $t = 1$ and **ii** $t = 3$.
b What is the average rate of change between $t = 1$ and $t = 3$?
c Find the instantaneous rate when:
i $t = 0$ **ii** $t = 1$ **iii** $t = 3$ **iv** $t = 4$.
d Why is the rate increasing in the first 4 hours?

9 The weight, W kg, of a foal at any time, t weeks, after birth is given by:

$$W = 80 + 12t - \tfrac{3}{10}t^2 \quad \text{where} \quad 0 < t < 20$$

a What is the weight of the foal at birth?
b Find an expression for the rate of change of weight at any time, t.
c Find the rate of change after:
i 5 weeks **ii** 10 weeks **iii** 15 weeks.
d Is the rate of change of the foal's weight increasing or decreasing?
e When does the foal weigh 200 kg?

10 The weekly profit, P (hundreds of dollars), of a factory is given by $P = 4.5n - n^{\frac{3}{2}}$, where n is the number of employees.
a Find $\dfrac{dP}{dn}$.
b Hence, find the rate of change of profit, in dollars per employee, if the number of employees is:
i 4
ii 16
iii 25.
c Find n when the rate of change is zero.

11 Gas is escaping from a cylinder so that its volume, V cm^3, t seconds after the leak starts, is described by $V = 2000 - 20t - \frac{1}{100}t^2$.
a Find the rate of change after:
i 10 seconds
ii 50 seconds
iii 100 seconds.
b Is the rate of change ever positive? Why?

12 Assume an oil spill from an oil tanker is circular and remains that way.
a Write down a relationship between the area of the spill, A m^2, and the radius, r metres.
b Find the rate of change of A with respect to the radius, r.
c Find the rate of change of A when the radius is:
i 10 m
ii 50 m
iii 100 m.
d Is the area increasing more rapidly as the radius increases? Why?

13 A spherical balloon is being inflated.

a Express the volume of the balloon, V m^3, as a function of the radius, r metres.

b Find the rate of change of V with respect to r.

c Find the rate of change when the radius is:

i 0.1 m

ii 0.2 m

iii 0.3 m.

14 A rectangular fish tank has a square base with its height being equal to half its base length.

a Express the length and width of the base in terms of its height, h.

b Hence, express the volume, V m^3, in terms of the height, h, only.

c Find the rate of change of V when:

i $h = 1$ m

ii $h = 2$ m

iii $h = 3$ m.

15 For the triangular package shown find:

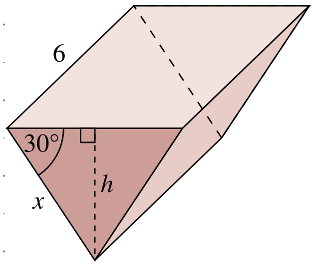
a x in terms of h

b the volume, V, as a function of h only

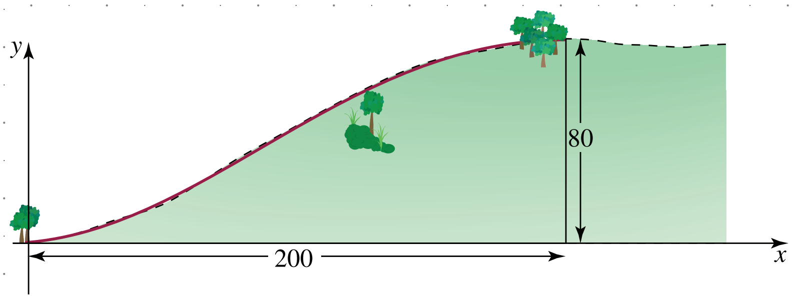
c the rate of change of V when

i $h = 0.5$ m

ii $h = 1$ m.



16 A new estate is to be established on the side of a hill.



Regulations will not allow houses to be built on slopes where the gradient is greater than 0.45. If the equation of the cross-section of the hill is

$$y = -0.000\,02x^3 + 0.006x^2$$

find:

a the gradient of the slope $\dfrac{dy}{dx}$

b the gradient of the slope when x equals:

i 160

ii 100

iii 40

iv 20

c the values of x where the gradient is 0.45

d the range of heights for which houses cannot be built on the hill.

17 A bushfire burns out A hectares of land, t hours after it started according to the rule
$$A = 90t^2 - 3t^3$$
a At what rate, in hectares per hour, is the fire spreading at any time, t?
b What is the rate when t equals:
i 0 **ii** 4 **iii** 8 **iv** 10 **v** 12 **vi** 16 **vii** 20?
c Briefly explain how the rate of burning changes during the first 20 hours.
d Why isn't there a negative rate of change in the first 20 hours?
e What happens after 20 hours?
f After how long is the rate of change equal to 756 hectares per hour?

CHAPTER 9 Applications of differentiation

Exercise 9A — Rates of change

1 a 13 **b** $f'(x) = 2x + 5$ **c** $f'(5) = 15$
2 a i $V = 0$ cm^3 **ii** $V = 800$ cm^3
b 80 cm^3/s
c i 0 cm^3/s **ii** 120 cm^3/s **iii** 0 cm^3/s
3 E **4** C **5** C
6 a $h'(t) = 18 - 6t$
b i 6 m/s **ii** 0 m/s **iii** –6 m/s
c The ball stops rising, that is, it reaches its highest point.
d 12 m/s
7 a $\dfrac{dx}{dt} = -4t + 40$ **b i** 20 m/s **ii** 4 m/s **iii** –4 m/s
c The lift changed direction.
d $t = 10$ s and $x = 200$ m
8 a i 4000 **ii** 15 000
b 5500 people per hour
c i 3500 people/hour **ii** 4500 people/hour
iii 6500 people/hour **iv** 7500 people/hour
d More people arrive closer to starting time.
9 a 80 kg **b** $\dfrac{dW}{dt} = 12 - 0.6t$
c i 9 kg/week **ii** 6 kg/week **iii** 3 kg/week
d Decreasing **e** 20 weeks
10 a $\dfrac{dP}{dn} = 4.5 - 1.5n^{\frac{1}{2}}$
b i \$37.50 **ii** –\$9.38 **iii** –\$12.00 **c** $n = 9$
11 a i –20.2 cm^3/s **ii** –21 cm^3/s **iii** –22 cm^3/s
b No, because the volume is always decreasing.
12 a $A = \pi r^2$ **b** $\dfrac{dA}{dr} = 2\pi r$
c i 20π m^2/m **ii** 100π m^2/m **iii** 200π m^2/m
d Yes, because $\dfrac{dA}{dr}$ is increasing.
13 a $V = \frac{4}{3}\pi r^3$ **b** $\dfrac{dV}{dr} = 4\pi r^2$
c i 0.04π m^3/m or 0.13 m^3/m
ii 0.16π m^3/m or 0.50 m^3/m
iii 0.36π m^3/m or 1.13 m^3/m
14 a Length = $2h$, width = $2h$
b $V = 4h^3$
c i 12 m^3/m **ii** 48 m^3/m **iii** 108 m^3/m
15 a $x = 2h$ **b** $V = 6\sqrt{3}h^2$
c i $\dfrac{dV}{dh} = 6\sqrt{3}$ **ii** $\dfrac{dV}{dh} = 12\sqrt{3}$
16 a $\dfrac{dy}{dx} = -0.000\,06x^2 + 0.012x$
b i 0.384 **ii** 0.6 **iii** 0.384 **iv** 0.216
c $x = 50$ and $x = 150$ **d** $12.5 < y < 67.5$
17 a $\dfrac{dA}{dt} = 180t - 9t^2$ hectares/hour
b i 0 **ii** 566 **iii** 864 **iv** 900 **v** 864 **vi** 576
vii 0 (all hectares/hour)
c The fire spreads at an increasing rate in the first 10 hours, then at a decreasing rate in the next 10 hours.
d The fire is spreading, the area burnt out by a fire does not decrease.
e The fire stops spreading; that is, the fire is put out or contained to the area already burnt.
f $t = 6$ and $t = 14$ hours.

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$

$S = \frac{a}{1 - r}, \; |r| < 1$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

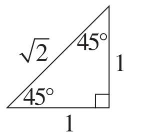
$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

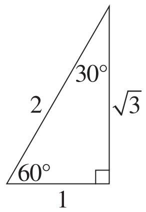
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

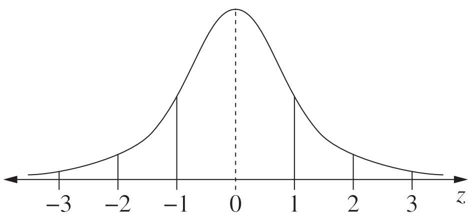
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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